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The entry of a falling film into a pool and the air-entrainment problem

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When a film of liquid runs down a vertical wall and enters a large pool a number of stationary horizontal ripples are observed on the film just above the point of entry. These ripples are explained and analysed by the method of matched expansions and the results have a bearing on more-complicated problems. Next the case when the wall itself is moving vertically downwards is analysed. This is of interest in roll-coating technology, where it is normally desired to achieve smooth entry of the descending film, without air entrainment. The paper gives sufficient conditions for smooth entry, and these are consistent with known experimental results.

1. Introduction

Suppose a thin liquid film of uniform thickness flows down a vertical wall and enters a large pool. The free surface does not simply turn through a right-angled bend but exhibits several horizontal ripples near the point of entry (see figure 1). These ripples have been described by Cook & Clarke (1973) and by Cullen & Davidson (1957), for example. The explanation, in terms of the interplay between gravity, friction and capillarity, was given in essence by Ruschak (1978), and will be amplified somewhat below.

However Ruschak gave only numerical solutions of the ordinary differential equation for the film thickness, and, although he gave an estimate for the film thickness just before entry into the pool which is asymptotically correct (in a certain limit to be explained later), the explanation is not absolutely clear and he was not able to use this to construct an asymptotic solution of the differential equation.

It seems worthwhile to pursue this problem further. To begin with, this homely piece of fluid mechanics is interesting in its own right; but it seems furthermore to be the simplest manifestation of a problem in matched expansions which is at the root of difficulties encountered elsewhere when a thin film with a free surface approaches a stagnant or hydrostatic region (Bretherton 1961; Jones & Wilson 1978). We therefore give below the solution of Ruschak's problem by matched expansions.

We also consider a natural extension of this problem in which the wall itself is moving vertically downwards. The case of motion vertically upwards amounts to the drag-out problem (Wilson 1982), in which the film thickness at large heights above the pool is to be determined. In the present case the thickness can be specified independently of the velocity of the wall so that a two-parameter problem emerges. The flow arises in connection with roll-coating processes. Here a roller is arranged, with its axis horizontal, half-submerged in a bath of liquid and caused to rotate. The amount of liquid picked up is given by the solution of the drag-out problem. Some of this liquid is removed and the remainder re-enters the bath in the manner to be analysed here. In some cases a smooth entry is observed, but in others the free surface of the pool dips and air may be entrained. Experiments on these lines have been reported by Bolton & Middleman (1980) and by Wilkinson (1975). Although we shall not obtain here any solutions of the differential equations corresponding to air entrainment (they all exhibit smooth entry), some useful conclusions can nonetheless be drawn.

2. Equations of motion

Making the usual approximations of lubrication theory, and referring to figure 1, we have

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial \bar{x}}+\frac{\mu}{\rho}\frac{\partial^2 u}{\partial \bar{y}^2}+g, \tag{1}$$

$$0=-\frac{1}{\rho}\frac{\partial p}{\partial \bar{y}}, \tag{2}$$

and the boundary conditions are

$$\begin{gathered} u=U \quad \text{on} \quad \bar{y}=0, \\ \left.\begin{aligned} \frac{\partial u}{\partial \bar{y}}&=0 \\ p+T\kappa&=0 \end{aligned}\right\} \quad \text{on} \quad \bar{y}=\bar{h}. \end{gathered} \tag{3}$$

Here T is the surface tension, assumed constant, and κ, the free-surface curvature, is related to $\bar{h}$ by $\kappa=\bar{h}_{\bar{x}\bar{x}}(1+\bar{h}_{\bar{x}}^2)^{-\frac{3}{2}}$. It is necessary to prescribe the film thickness at $\bar{x}=-\infty$, say $\bar{h}=h_0$; the surface is assumed to be free of waves sufficiently far above the pool.

It is a simple matter to derive a differential equation for $\bar{h}(\bar{x})$; we use (2) and (3) to calculate the pressure gradient term in (1) in terms of κ, and hence of $\bar{h}$, and then integrate (1) three times with respect to $\bar{y}$ to obtain the flux down the wall. This quantity is, of course, a constant, and further must have the value $Uh_0+\rho g h_0^3/3\mu$ (volume per unit time per unit length of wall) given the earlier assumption that the film has a uniform thickness at large distances from the pool. A convenient dimensionless version of the final equation is

$$\phi^3\frac{d}{dx}\left\{\frac{\phi_{xx}}{(1+\epsilon^2\phi_x^2)^{\frac{3}{2}}}\right\}=1-\phi^3+3\sigma^3(1-\phi). \tag{4}$$

As a preliminary to explaining this equation we note that three lengthscales occur naturally, namely h_0 and two others denoted D and d and given by

$$D=\left(\frac{T}{\rho g}\right)^{\frac{1}{2}}, \quad d=\left(\frac{\mu U}{\rho g}\right)^{\frac{1}{2}}. \tag{5}$$

Now D is the lengthscale of the hydrostatic meniscus, and d is the order of magnitude of film thickness at which the motion of the wall and the force due to gravity produce comparable velocities in the film.

In (4) we have

$$\phi=\frac{h}{h_0}, \quad x=\frac{\bar{x}}{l}, \tag{6}$$

where the first of these is forced, and the scale for $\bar{x}$, namely l, is yet another length given by

$$l=\left(\frac{Th_0}{\rho g}\right)^{\frac{1}{3}} \tag{7}$$

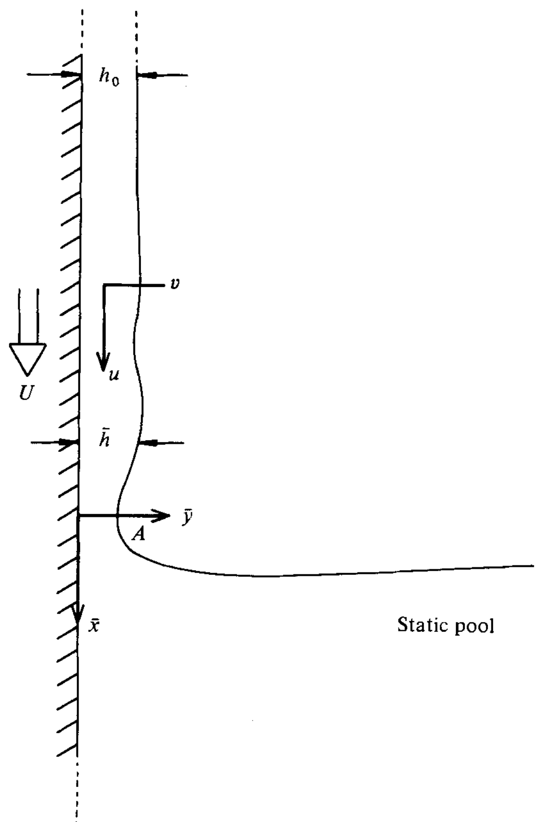

FIGURE 1. Sketch of coordinate system. A denotes the final wave trough.

and is determined by matching up the surface-tension and gravity terms in (4). The two dimensionless parameters ϵ and σ are defined by

$$\epsilon^3 = \frac{\rho g h_0^2}{T} = \frac{h_0^2}{D^2},$$
$$\sigma^3 = \frac{\mu U}{\rho g h_0^2} = \frac{d^2}{h_0^2}. \tag{8}$$

The analysis will proceed throughout on the assumption that ϵ is small. Indeed, this statement has already been used implicitly when writing down the original equations, since the inequality $h_0 \ll D$ is the basis of the boundary-layer approximation. This allows us to discard such terms as $\partial^2 u/\partial \bar{x}^2$ from (1), this term being smaller than $\partial^2 u/\partial \bar{y}^2$ by ϵ^2. It will be noted, however, that one term $O(\epsilon^2)$ has been retained in our basic equation, (4). This is because this term cannot be uniformly neglected in the problem. In the meniscus region, where the film meets the pool, ϕ_x takes values $O(\epsilon^{-1})$, forcing us to retain this term in our equation. Conversely, it is easy to verify *a posteriori* that the terms that we have omitted are relatively small in all regions of the flow.

We shall try to allow as general a range as possible for σ. However, there is another basic restriction embodied in the approximations that produced (3). If the viscous stresses at the free surface are to be small in comparison with surface tension, then the capillary number $\mathscr{C}$ must be small. This is given by

$$\mathscr{C} = \frac{\mu U}{T} = \frac{d^2}{D^2} = \sigma^3 \epsilon^3, \tag{9}$$

and so σ cannot be allowed to be as large as ϵ^{-1}.

A similar analysis has been carried out by Wilson (1982) for the drag-out problem. The scalings here are different and have been determined so that the motion of the wall, measured relative to gravity (as it were) by the parameter σ, can be conveniently thought of as a perturbation of the entry problem considered by Ruschak (1978), for which $\sigma = 0$, of course. (This perturbation need not in fact be 'small'.)

To complete the determination of the appropriate solution of (4) we have the condition

$$\phi \to 1 \quad \text{as} \quad x \to -\infty,$$

and also the requirement that the solution must merge somehow into a hydrostatic meniscus. How this can be accomplished is in fact the technical point of the paper and will be considered in detail in §§3 and 4.

3. Fixed wall

When the wall is fixed σ takes the value zero. If we also follow the natural procedure for small ϵ and omit the term in ϵ from (4), then the equation becomes

$$\phi_{xxx} = \frac{1}{\phi^3} - 1, \tag{10}$$

a form that was considered by Jones & Wilson (1978). There are solutions that satisfy the boundary condition $\phi \to 1$ as $x \to -\infty$, but there is no simple way of matching them forward into the meniscus region. To show the problem we need to describe the nature of these solutions in more detail. At large negative x, ϕ approaches unity, with the asymptotic difference between ϕ and unity being a harmonic oscillation with amplitude that decays exponentially. This oscillation grows in the positive direction until its amplitude is comparable to unity, whereupon it loses its harmonic character and develops into a nonlinear oscillation which dominates the solution. These oscillations are better described as a series of ever-increasing 'leaps'; a leap consisting of a long region in which ϕ takes large values and the gradient changes slowly from positive to negative, followed by a short region in which ϕ is less than unity and the gradient changes rapidly from negative to positive. In the region where ϕ is large, (10) is approximately $\phi_{xxx} = -1$, so that, for a leap starting from near zero at $x = x_0$,

$$\phi \approx -\tfrac{1}{6}(x-x_0)^3 + a(x-x_0)^2 + b(x-x_0).$$

In a typical leap the gradients are large at the start, so that the coefficients a and b are large. Eventually, however, the cubic term must dominate and the solution will be brought back to nearly zero, but now with negative curvature and gradient. However, once ϕ becomes small, (10) is approximately $\phi_{xxx} = 1/\phi^3$, so that the rate of change of curvature is large and positive. In fact the term on the right-hand side is sufficiently strong to prevent ϕ ever reaching zero, and to reverse the signs of the curvature and gradient over a very short region. Further, it amplifies the magnitude of these quantities so that the next leap is started with even larger values of a and b. The result is that ϕ attains even greater values than before, and takes yet longer before it is brought back towards small values again.

Now to match into the meniscus region ϕ must increase formally by an order of magnitude; namely, as we show presently, $\epsilon^{-\frac{3}{2}}$. Since we are considering the limit $\epsilon \rightarrow 0$ this means that ϕ must increase by an indefinitely large amount. However, such an increase is not possible over a finite region. Equation (10) is not singular at all, and so, if we imagine integrating forward from some finite starting value of ϕ, the final value reached, however amplified by the nonlinear oscillations between, is still finite and must therefore be formally considered as still being of order unity. Thus to achieve an indefinitely large increase in ϕ it is necessary for the solution to first undergo an indefinitely large number of nonlinear leaps.

This conclusion is at first hard to accept since it seems to contradict both experiment and the numerical solutions with fixed small ϵ given by Ruschak (1978), which show not an oscillatory approach to infinity but a monotonic approach after only one or two nonlinear oscillations. The first of these features, the monotonic behaviour, is easy to reconcile. Suppose that we have already traversed a formally infinite number of oscillations so that ϕ has already reached an asymptotically large value. Then, if we integrate forward one further step from one crest to the next, our starting conditions must be expressed as a suitable function of ϵ that reflects the asymptotic value already achieved. The fact that the parameter ϵ now appears in the basic formulation of this problem means that there is now the possibility that the solution becomes asymptotically amplified as the trough is traversed, and this is, in fact, what happens. There is, then, a *final* trough, denoted by A in figure 1, across which ϕ increases in order of magnitude to reach the desired total order of magnitude change of $\epsilon^{-\frac{3}{2}}$. After this change the solution can then depart uniformly to infinity.

Two things remain: we must verify the above description by constructing an asymptotic solution which has these features, and then we must show why a large number of nonlinear oscillations are not observed in practice. We begin the construction by examining the final trough. A new scale is necessary for ϕ, which in this region is small in terms of ϵ. Also, to achieve a balance in the equation the coordinate x must be rescaled. We therefore introduce new variables into (4) to describe the surface near A, namely ψ_1 and X_1, where

$$\phi = \alpha\psi_1, \quad x = \beta X_1, \tag{11}$$

and α and β are small parameters to be determined. (It may be helpful at this point to refer to figure 2, where the asymptotic structure of the various regions near A is sketched. The orders of magnitude indicated there are to be established in the analysis that now follows.) To balance the equation we require

$$\alpha^4 = \beta^3 \tag{12}$$

so that the equation itself reduces to

$$\psi_1^3 \frac{d^3\psi_1}{dX_1^3} = 1. \tag{13}$$

In order that the matchings of this trough region to the regions on either side of it produce an overall order of magnitude increase in the positive X_1 direction, the solution must have different coordinate behaviour at plus and minus infinity, increasing more rapidly as $X_1 \rightarrow +\infty$ than as $X_1 \rightarrow -\infty$. Since the solution for ψ_1 large is either quadratic or linear in X_1, we clearly require that $\psi_1 = O(X_1^2)$ as $X_1 \rightarrow +\infty$ but that $\psi_1 = O(X_1)$ as $X_1 \rightarrow -\infty$. The constants α and β are now determined by the requirement that this solution can be matched forward into the meniscus region. In

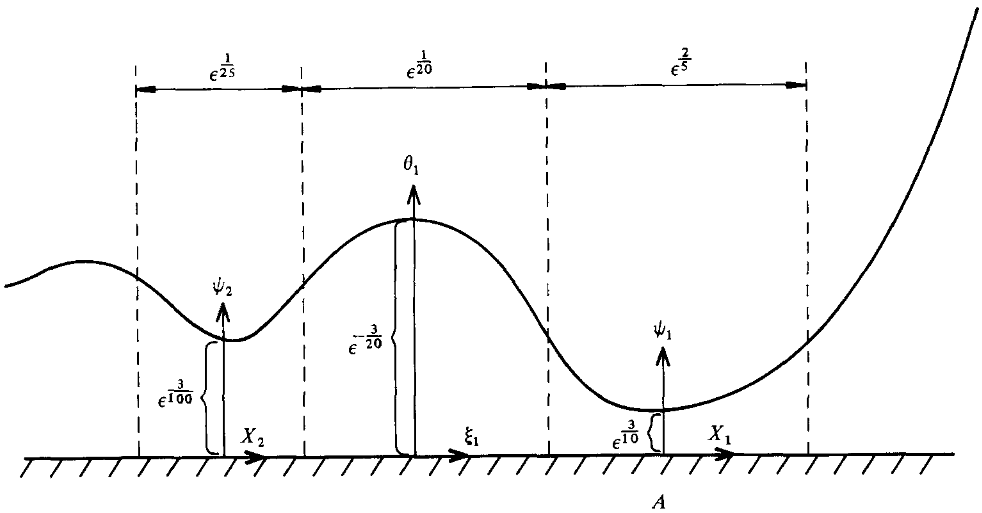

FIGURE 2. Asymptotic structure of the solution near the final wave trough A indicated schematically. The lengthscales are in terms of the basic order-unity variables x and ϕ.

that region the correct scales for both $\bar{h}$ and $\bar{x}$ are well known, both being equal to D, so, if we denote the corresponding dimensionless variables by Φ and X, then we must have

$$\Phi = \frac{\bar{h}}{D} = \epsilon^{\frac{3}{2}}\phi,$$

$$X = \frac{\bar{x}}{D} = \epsilon^{\frac{1}{2}}x. \tag{14}$$

With these scalings Φ satisfies the hydrostatic equation

$$\frac{d}{dX}\left\{\frac{\Phi_{XX}}{(1+\Phi_X^2)^{\frac{3}{2}}}\right\} = -1 \tag{15}$$

to leading order in ϵ. This static meniscus must approach the vertical wall with apparent contact angle zero to give the required quadratic behaviour near the wall, as in the drag-out problem. Solving this outer problem we find, in fact, that

$$\Phi \sim X^2/2^{\frac{1}{2}} \quad \text{as} \quad X \to 0. \tag{16}$$

For the scalings (11) and (14) to be compatible with these quadratic forms for ψ_1 and Φ it is necessary that

$$\beta^2 = \alpha\epsilon^{\frac{1}{2}}, \tag{17}$$

and combining this with (12) we can now determine the trough scalings to be

$$\beta = \epsilon^{\frac{2}{5}}, \quad \alpha = \epsilon^{\frac{3}{10}}. \tag{18}$$

The value of α was actually predicted by Ruschak. He examined a sequence of curves computed numerically with different values of ϵ, and by assuming that the minimum of the troughs had a power-law behaviour was able to evaluate the missing exponent. However, he was not able to fit this result into any comprehensive scheme.

With these values for α and β, the forward matching reduces to the boundary condition $\psi_1 \sim X_1^2/2^{\frac{1}{2}}$ as $X_1 \to \infty$, and, with the additional condition $\psi_1 \sim -aX_1$ as $X_1 \to -\infty$, the solution of (13) is fully determined. The numerical solution given by

Jones & Wilson (1978) can be used to determine the constant a and the minimum value of ψ_1. We have, in fact,

$$a \approx 1{\cdot}03,$$

$$\psi_1(\min) \approx 1{\cdot}14,$$

and this minimum value is in good agreement with Ruschak's (1978) numerical results. (Some care is necessary when making the comparison because the scalings are different. The agreement was perfect to within the accuracy that figures can be read off such a small graph.)

Next we must explain how the solution for ψ_1 can be matched upstream into a solution of (10). The equation for ψ_1, when the small terms are restored, is

$$\psi_1^3 \frac{d}{dX_1}\left\{\frac{\psi_1''}{(1+\epsilon^{\frac{9}{5}}\psi_1'^2)^{\frac{3}{2}}}\right\} = 1-\epsilon^{\frac{9}{10}}\psi_1^3. \tag{19}$$

As we leave the neighbourhood of A in the upstream direction (i.e. as $X_1 \to -\infty$) the solution grows linearly, as noted, and will continue to do so until the small term on the right-hand side of (19) enters and turns it round. It is easy to verify that this turn around has not been achieved by the time ψ_1 is $O(\epsilon^{-\frac{3}{10}})$, i.e. when ϕ is $O(1)$. Thus, as has been remarked already, it is not possible for a direct match to be achieved between ψ_1 and (10). Instead, the solution will overshoot order-unity values and thus enter a region where it is asymptotically large, although not as large as it was in the meniscus region. In this region the solution has the same behaviour as a crest of the nonlinear 'leaps' described earlier. Following it in the negative x-direction, the solution will increase to a maximum value and then decrease towards zero. When it becomes sufficiently small, it enters a trough region where it becomes turned around once more and is directed outwards towards yet another crest, whereupon the pattern repeats itself. In passing through this second trough, and indeed through all subsequent troughs, it undergoes a further reduction in its order of magnitude. However, the sequence of these reductions is such that values formally of order unity can never actually be reached, so that an infinite series of troughs and crests must be traversed before the solution of (10) can be approached.

To demonstrate this progress of events we shall describe the first crest and its subsequent trough in detail, and then infer the continuation. We must introduce freshly scaled variables for the crest region, so we introduce a small parameter λ, to be determined, and put

$$\theta_1 = \lambda\psi_1, \quad \xi_1 = \lambda X_1, \tag{20}$$

since the linear behaviour of ψ_1 implies that the same parameter must occur in both equations. The required balance in (19) is then achieved with the choice $\lambda = \epsilon^{\frac{9}{20}}$, and the equation satisfied by θ_1 is

$$\theta_1''' = -1, \tag{21}$$

to leading order. When θ_1 is order unity, ϕ is $O(\epsilon^{-\frac{3}{20}})$, and because this is formally infinite it is not possible to match directly with the ultimate upstream condition $\phi = 1$. Instead the solution will reach a maximum and return to small values, thus entering the second trough region. This region is very similar to the trough at A, and it can be verified *a posteriori* that its governing equation has the same form as (13). In that case, if the solution is to undergo the desired reduction in magnitude as it passes through the trough, it must match into the crest on its positive side with quadratic behaviour, and into the crest on its negative side with linear behaviour. Then the analysis will be analogous to that of (13), where an order-of-magnitude reduction was also found.

The quadratic matching requirement places a condition on the function θ_1, namely that it must have a double zero at some unknown value of ξ_1, $\xi_1 = \xi_0$ say. The matching condition with the trough at A provides the additional conditions

$$\theta_1 = 0, \quad \theta_1' = -a \quad \text{at} \quad \xi_1 = 0, \tag{22}$$

where we have also used this condition to define the otherwise arbitrary origin of ξ_1. All these conditions make θ_1 fully determinate and it can be calculated that it takes a maximum value of $(\frac{32}{243}a^3)^{\frac{1}{2}}$, while ξ_0 turns out to be $-(6a)^{\frac{1}{2}}$.

With the crest region solved we can move on to the second trough. If ψ_2 and X_2 are appropriate $O(1)$ variables in this region then we must introduce the parameter γ, to be determined, and put

$$\psi_2 = \gamma^2\theta_1, \quad X_2 = \gamma(\xi_1 - \xi_0),$$

since these relations allow quadratic matching between the functions ψ_2 and θ_1. To balance terms in our basic equation we must then choose $\gamma = \epsilon^{-\frac{9}{100}}$, whereupon (13) is obtained, except that the subscripts are different. Completing the details of the matching then yields the conditions

$$\psi_2 \sim \left(\frac{a}{6}\right)^{\frac{1}{2}} X_2^2 \quad \text{as} \quad X_2 \to \infty,$$

$$\psi_2 = O(X_2) \quad \text{as} \quad X_2 \to -\infty,$$

and these make the function ψ_2 fully determinate, except for an arbitrary origin shift. The solution can be found numerically if required.

The important feature to concentrate attention on here, however, is the order-of-magnitude changes that have occurred. The value $\gamma = \epsilon^{-\frac{9}{100}}$ corresponds to ϕ being $O(\epsilon^{\frac{3}{100}})$ in this trough region, as compared with being $O(\epsilon^{\frac{3}{10}})$ near A. And if we were to continue our analysis to the infinite series of troughs and crests extending upstream from A we would find that the nth trough (counting A as $n = 1$) is always small, with ϕ of order $\epsilon^{3/10^n}$, while the nth crest is always large with ϕ of order $\epsilon^{-3/2.10^n}$. However, although these numbers are always *formally* infinitesimal or infinite respectively, this is a very rapidly changing sequence, and for any reasonable value of ϵ they are all (except for the first) as near unity as makes no practical difference. For instance, with the impossibly small value $\epsilon = 10^{-6}$ (with $D = 1$ cm this value would require h_0 to be less than an atomic diameter), $\epsilon^{\frac{3}{10}} \approx 0{\cdot}16$ and $\epsilon^{\frac{3}{100}} \approx 0{\cdot}66$. Thus, heuristically, the infinite wave train flattens itself out and becomes order one after only one or two oscillations. This is why only one or two waves are seen in the experiments and in the numerical work.

Finally, we outline briefly the physical explanation of the waves. At any point where the free surface is concave to the wall, the pressure is reduced by surface tension, and the fluid there will be moving somewhat faster than immediately upstream because of the pressure-gradient force. By continuity the film must therefore be thinner. Thus at points where the free surface is concave to the wall the film is thinner than average, and similarly where it is convex it is thicker, producing a wavy effect. Nevertheless, the free surface can tend to infinity (in these coordinates) and merge with the horizontal free surface of the pool. This happens when the gradient becomes so large that the curvature can no longer be approximated adequately by the second derivative. We can see from (15) that the surface can have constant negative rate of change of curvature, but never reach a maximum and turn round.

4. Moving wall

We now turn to the various possibilities that arise when $\sigma > 0$. It is convenient to imagine that σ has some definite order of magnitude in terms of ϵ, say $\sigma = \epsilon^{-k}$. Now it is easy to see that, if σ is order 1, that is $k = 0$, the analysis of § 3 goes through with only trivial changes to the algebra, and if σ is small, i.e. $k < 0$, we have a trivial regular perturbation. We may therefore confine our attention to the cases when σ is large. In view of (9) we need, or may, only consider the range $0 < k < 1$.

When σ is formally large it is necessary first to rescale x in (4). We put

$$\hat{x} = 3^{\frac{1}{3}}\sigma x,$$

and then (4) becomes

$$\phi^3 \frac{d}{d\hat{x}}\left\{\frac{\phi''}{(1+3^{\frac{2}{3}}\sigma^2\epsilon^2\phi'^2)^{\frac{3}{2}}}\right\} = \frac{1}{3\sigma^3}(1-\phi^3)+1-\phi. \tag{23}$$

So the first task is to consider the first approximation to (23), namely

$$\phi''' = \frac{1}{\phi^3} - \frac{1}{\phi^2}. \tag{24}$$

This can be made to satisfy the condition $\phi \to 1$ as $\hat{x} \to -\infty$, and at $-\infty$ we have the asymptotic form

$$\phi \sim 1 + C\exp(\tfrac{1}{2}\hat{x})\cos(\tfrac{1}{2}\sqrt{3}\,\hat{x})\ldots. \tag{25}$$

Here the free choice of origin of $\hat{x}$ has been used to eliminate a solution with the opposite phase. Meanwhile, at $+\infty$ we have

$$\phi \sim B\hat{x}^2. \tag{26}$$

Numerical integrations indicate that any positive value of B can be made to occur by a suitable choice of the constant C. The relation between them is sketched in figure 3. Note that in view of (25) it is necessary only to consider a range of C such as $(C, C\exp(-2\sqrt{\tfrac{1}{3}}\,\pi))$, because outside this we get the same solutions starting at a different place. The range of C shown in the figure arose because the integrations were started at $\hat{x} = 0$. (Further terms in the expansion (25) were incorporated into the numerical work to improve the accuracy.) The values that are actually required will emerge from the process of matching (26) to the meniscus, and we turn now to this question.

On examination of (23) we see that when ϕ and ϕ' become large there are two possible non-uniformities, according to which of the small terms (in the curvature, on the left, or in the gravitational term, on the right) enters first. A special case is distinguished, when these small terms enter simultaneously, which divides two asymptotically different regimes, and it is convenient to consider this first. We put

$$\tilde{\phi} = \alpha\phi, \quad \tilde{x} = \beta\hat{x}, \tag{27}$$

where α and β are small parameters to be determined. We are supposing that the constant B in (26) is a fixed order-unity number for the present, although this will be relaxed later. In view of this quadratic growth it will be necessary, if ϕ and $\tilde{\phi}$ are to match, that

$$\alpha = \beta^2. \tag{28}$$

For the two small terms in (23) to increase to order unity under the transformation (27) we must have

$$\frac{\sigma\epsilon\beta}{\alpha} = 1, \quad \frac{\beta^2}{\alpha^4} = \frac{1}{\sigma^3\alpha^3}, \tag{29}$$

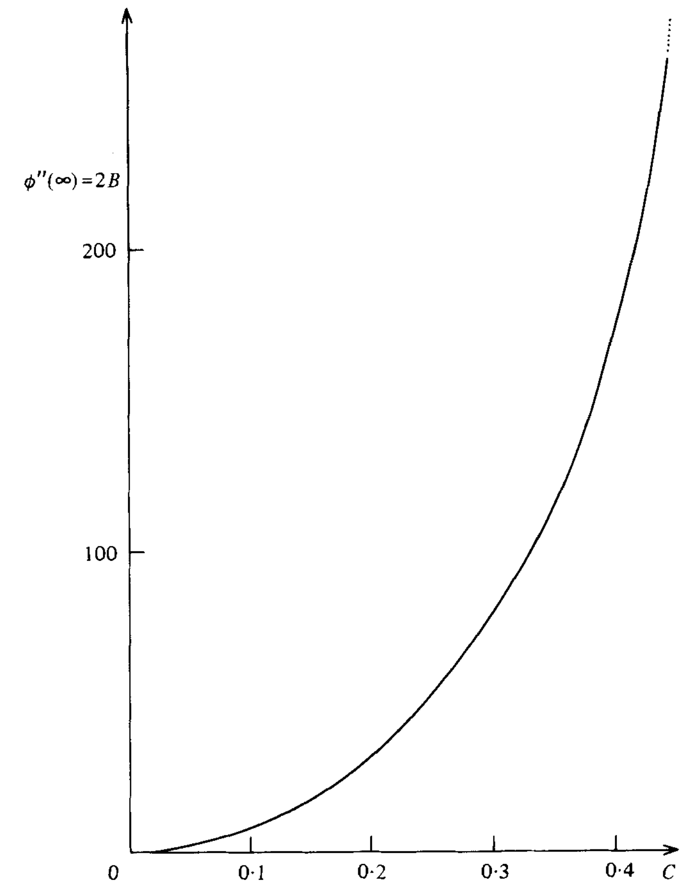

FIGURE 3. Relation between the constants B and C of (25) and (26). B appears to vanish when $C = 0{\cdot}0120$, approximately, and tends to infinity at $C = 0{\cdot}451$, approximately, the ratio of these two values being about $\exp(-2\sqrt{\tfrac{1}{3}}\pi)$, as explained in the text. The integrations were troublesome for large values of B and no great effort was made to obtain very accurate results.

and the solution of (28) and (29) is

$$\alpha = \epsilon^{\frac{3}{2}}, \quad \beta = \epsilon^{\frac{3}{4}}, \quad \sigma = \epsilon^{-\frac{1}{4}}. \tag{30}$$

Now we can see easily that the scales for $\bar{h}$ and $\bar{x}$ have increased to D, so that $\tilde{\phi}$ can be identified with Φ and $\tilde{x}$ with X, defined in (14), and the static meniscus region has been reached. We may put

$$\sigma = p\epsilon^{-\frac{1}{4}},$$

where p is a fixed order-unity constant, and replace (27) by

$$\Phi = \epsilon^{\frac{3}{2}}\phi, \quad X = 3^{-\frac{1}{3}}p^{-1}\epsilon^{\frac{3}{4}}\hat{x}. \tag{31}$$

Here the factor $3^{-\frac{1}{3}}p^{-1}$ has been inserted so as to give the correct meniscus equation, (15), free of extraneous factors of p, when (23) is transformed. Finally, matching (26) and (16) gives

$$Bp^2 = 2^{-\frac{1}{2}}3^{-\frac{2}{3}}, \tag{32}$$

which fixes B, and hence C, and thus the complete solution.

The case just considered corresponds to $k = \frac{1}{4}$, and we now turn to the range $0 < k < \frac{1}{4}$. An examination of (23) shows that as ϕ increases according to (26) the gravitational term ϕ^3/σ^3 on the right enters before the small term $\sigma^2\epsilon^2\phi'^2$ in the curvature. The equation satisfied by $\tilde{\phi}$ now takes the form

$$\tilde{\phi}''' = -1 \tag{33}$$

to leading order, so the solution turns round and returns to small values (rather like the function ϕ_1 of §3). The solution can only escape to infinity to match with the static meniscus if it first takes on formally small values, again as in §3. Repeating that analysis we make the transformation (11), and we obtain (12) and (13), but (17) is replaced by

$$\beta^2 = \alpha\epsilon^{\frac{1}{2}}\sigma^2, \tag{34}$$

and the solution is

$$\alpha = \epsilon^{\frac{3}{10}}\sigma^{\frac{6}{5}}, \quad \beta = \epsilon^{\frac{2}{5}}\sigma^{\frac{8}{5}}. \tag{35}$$

We see that the order of magnitude of ϕ, which is α, increases from $\epsilon^{\frac{3}{10}}$, when σ is order unity, to order unity when σ is order $\epsilon^{-\frac{1}{4}}$. This means that waves diminish in amplitude as σ increases.

Finally we consider the range $\frac{1}{4} < k < 1$. Now we have to abandon the idea that the constant B in (26) is order unity which leads to a matching problem with no solution. (Briefly, we should find that the small curvature term in (23) enters before the gravitational term, and as a consequence the surface would have to have constant curvature, i.e. be an arc of a circle. This cannot match to a static meniscus.) The correct form of the solution can be deduced simply by letting p tend to infinity in the analysis given earlier. Then we see from (32) that the constant B must be small, and in fact by arguments exactly similar to those leading to (32) we can show that

$$B = 3^{-\frac{2}{3}}2^{-\frac{1}{2}}\sigma^{-2}\epsilon^{-\frac{1}{2}}. \tag{36}$$

5. Concluding remarks

The fixed-wall case has been analysed by the method of matched expansions, and it turns out that an infinite number of overlapping regions is necessary, each of which contains a crest and a trough. The asymptotic sequence given by the successive film thickness in the troughs ($\epsilon^{3/10n}$, $n = 1, 2, \ldots$) is so feeble, however, that the terms are essentially unity after the first one or two, and the same applies to the crest heights.

The moving-wall case has been similarly analysed for the regime $\sigma \sim \epsilon^{-k}$, $0 \leqslant k < 1$. In the range $0 \leqslant k < \frac{1}{4}$ the behaviour is similar to the fixed-wall case, with an infinite number of regions being necessary. As k increases, the waves diminish in amplitude, reaching order unity when $k = \frac{1}{4}$, and also diminish in length (the wavelength is of order l/σ). These trends continue without qualitative change as k increases in the range $\frac{1}{4} < k < 1$. As k approaches 1, the approximations implicit in the basic equations (1)–(3) fail. First, the neglect of the viscous stress term in the last equation of (3) (which is of order $\mathscr{C} = \sigma^3\epsilon^3$ relative to the others) is no longer justified.† Secondly, the wavelength l/σ approaches $l\epsilon = h_0$, so that the neglect of x-derivatives in the viscous-stress term in (1) is no longer justified.

Within the limits explained, solutions have been obtained corresponding to smooth entry of the film into the pool. It must be admitted that the analysis was begun in the hope that the matching process would fail somehow (while the approximations remained valid) in a way which could have been interpreted as implying the onset of air entrainment. Nonetheless the results are of some use, because it is normally

† When $\sigma = 0$ this term is of relative order ϵ^3.

desired to avoid air entrainment, and the present analysis gives sufficient conditions for this, namely $\epsilon \ll 1$ and $\sigma\epsilon \ll 1$.

These conclusions are supported by the (admittedly sparse) experimental observations. Wilkinson (1975) gives the values of $\mathscr{C}$ at the onset of entrainment and these are all order unity (violating the condition $\mathscr{C} \ll 1$). There is not enough raw data to calculate ϵ but it was probably small. Bolton & Middleman also report the values of $\mathscr{C}$ at the onset of entrainment and most of these are in the range $1 < \mathscr{C} < 10$. Some are considerably smaller, but the evidence suggests that inertial effects were important in these cases – inertial effects, have, of course, been neglected throughout this paper. It seems likely that, when entrainment occurred and inertial effects were negligible, both σ and ϵ were of order unity.

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